Indian Statistical Institute
B.Math I Year
Second Semester Back Paper Examination, 2005-2006
Probability Theory II

Time: 3 hrs   Date: -06-06   Max. Marks : 100

1. Let $f(x,y) = C \ \exp\{-\frac{1}{2}(x^2 - xy + 4y^2)\}, (x,y) \in \mathbb{R}^2$.

   (a) What should C be so that f is a probability density function?

   b) Let (X,Y) be an $\mathbb{R}^2$-valued random variable with f above as its probability density function. Find the probability density functions of X and Y.

   c) For $y \in \mathbb{R}$, find the conditional probability density function of X given $Y = y$. [10+5+5]

2. Let X_1, X_2 be independent random variables each having $N(0,\sigma^2)$ distribution. Let $0 \leq \theta < 2\pi$. Define $Y_1 = X_1 \cos\theta - X_2 \sin\theta$, $Y_2 = X_1 \sin\theta + X_2 \cos\theta$. Find the joint probability density function of Y_1, Y_2. Also find $\mathrm{Cov}(Y_1, Y_2)$. [12+3]

3. Let X_1, X_2, X_3 be independent $N(0,1)$ random variables. Indicating clearly the results you are using, find the distribution of

   a) $X_1 + X_2 + X_3$, b) $X_1^2 + X_2^2 + X_3^2$

   c) $\frac{X_1^2+X_2^2}{2X_3^2}$, d) $\frac{X_2}{X_1}$ [5+5+5+5]

4. Let U_1, U_2 be independent random variables each having a uniform distribution over $(0,1)$. Find the probability density function of a) $U_1 + U_2$ b) $\max\{U_1, U_2\}$. [7+8]

5. For a random variable X, show that its characteristic function is real valued if and only if X and $(-X)$ have the same distribution function. [10]

6. Candidates A and B are contesting an election. Suppose 55% of the voters prefer candidate B. Using central limit theorem, find the probability that in a sample of size 100 at least one-half of those sampled will favour candidate A. [20]